IA-automorphisms of various classes of groups

Marcos Zyman

Borough of Manhattan Community College, CUNY

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Nilpotent Groups

- **Definition 1.** The *lower central series* of a group G is

$$\gamma_1(G) = G \geq \gamma_2(G) \geq \cdots \gamma_i(G) \geq \cdots$$

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where $\gamma_2(G) = G' = [G, G]$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$.

- **Definition 2.** A group G is termed *nilpotent* of class c if $\gamma_{c+1}(G) = 1$ for some c and $\gamma_c(G) \neq 1$.

IA-groups

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- The group of *IA-automorphisms* of G is $\ker(Aut(G) \mapsto Aut(G/G'))$. Thus, $IA(G) = \left\{\alpha \in Aut(G) : g^{-1}(g\alpha) \in G' \; (g \in G)\right\}.$

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- $G/\zeta(G) \cong Inn(G)$.

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- If G is torsion-free, $IA(G)$ is torsion-free.

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- A group G is *metabelian* if $G' = [G, G]$ is abelian.
- **Theorem** [MZ] If the p-localizations of two finitely generated nilpotent, torsion-free and metabelian groups are isomorphic, then the p-localizations of their IA-groups are also isomorphic.

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  1. If n is prime then $IA(G) = Inn(G)$.
  2. If n is square-free but not prime, then $IA(G)$ is finitely generated but $IA(G) \neq Inn(G)$.
  3. If n is not square-free then $IA(G)$ is not finitely generated.

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- The upper central series of G

$$1 = Z_0 \quad \trianglelefteq \quad Z_1 \quad \trianglelefteq \quad \ldots \quad \trianglelefteq \quad Z_i \quad \trianglelefteq \quad \ldots$$

is defined recursively for $i > 0$ by the condition

$$Z_{i+1}/Z_i = \zeta(G/Z_i).$$

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(i) $Z_1 = Z_2$.

(ii) $\zeta(G/\zeta(G)) = 1$.

(iii) $\zeta(G) = Z_i$ *for all* $i = 1, 2, \ldots$.

(iv) $Inn(G)$ *has trivial center.*

Results

- G will be termed an *$\mathcal{H}$-group* if $Z_1 = Z_2$ in the upper central series.

Lemma

Let G be an $\mathcal{H}$-group. Then

$$gp(Inn(G), Aut_c(G)) = Inn(G) \times Aut_c(G).$$

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- $Inn(G) \cap Aut_c(G) \leq \zeta(Inn(G)) = 1$.

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Lemma

Let G be an $\mathcal{H}$-group such that $G' \geq \zeta(G)$ and $IA(G/\zeta(G)) = Inn(G/\zeta(G))$. Then $Inn(G)$ is a complete set of representatives of $Aut_c(G)$ in $IA(G)$.

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$\overline{x}$ induces the desired inner automorphism. □

Results

Theorem

Let G be a group such that $Z_1 = Z_2$ in the upper central series and $\zeta(G) \leq G'$. If $IA(G/\zeta(G)) = Inn(G/\zeta(G))$, then

$$IA(G) = Inn(G) \times Aut_c(G).$$

A useful lemma

Lemma

Let $G = gp(x_1, \ldots, x_n)$ be a finitely generated $\mathcal{H}$-group such that $\zeta(G) \leq G'$. Suppose that there exists a presentation for G in which every relator is a product of commutators. If $Aut_c(G)$ is finitely generated, then $\zeta(G)$ is finitely generated.

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Proof.

Suppose $\{\alpha_1, \ldots, \alpha_m\}$ generates $Aut_c(G)$. Then

$$x_i\alpha_j = x_i z_{ij}$$

with $i = 1, \ldots, n$; $j = 1, \ldots, m$; and $z_{ij} \in \zeta(G)$. Then the z_{ij} generate the $\zeta(G)$. □

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- Hence, $Aut_c(G)$ is itself free abelian of infinite rank. $\Rightarrow$
- $IA(G)$ is not finitely generated.

Thank you!